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Space charge-limited emission studies using Coulomb's Law

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MONTEREY, CALIFORNIA

THESIS

SPACE CHARGE LIMITED EMISSION STUDIES USING COULOMB’S LAW

by

Christopher G. Carr

June 2004

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SPACE CHARGE LIMITED EMISSION STUDIES USING COULOMB'S LAW

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requirements for the degree of

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Child and Langmuir introduced a solution to space charge limited emission in an infinite area planar diode. The solution follows from starting with Poisson's equation, and requires solving a non-linear differential equation. This approach can also be applied to cylindrical and spherical geometries, but only for one-dimensional cases. By approaching the problem from Coulomb's law and applying the effect of an assumed charge distribution, it is possible to solve for space charge limited emission without solving a non-linear differential equation, and to limit the emission area to two-dimensional geometries. Using a Mathcad worksheet to evaluate Coulomb's law, it is possible to show correlation between the solution derived by Child and Langmuir and Coulomb's law.

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I. INTRODUCTION

A. APPLICATIONS OF VACUUM DIODES

Recent emphasis on missile defense and the expansion of electronic warfare has increased a need for high-energy electromagnetic weapons. Directed energy microwave weapons require high-energy electron beams to achieve the power levels adequate for standoff utilization, which can reach the gigawatt range for peak power, which is far beyond the power production of a semiconductor device.

One method of creating electron beams is through vacuum diodes. In a vacuum diode, an electron cloud is created at a cathode and accelerated across a vacuum gap to a collector anode. This can be done through thermionic emission, field effect emitters, or explosive plasma emitters. For the pulse power requirements of high power microwaves, explosive emitters ensure that enough electrons will be available to create currents necessary for high power devices within the desired turn-on time.

Experimental work on diode emitters is an ongoing subject, and is focused on diode geometries and materials. Suppression of plasma effects, cathode longevity, and the use of micro-fiber cathodes are all areas of interest in the creation of fieldable high-energy weapons. However, the electromagnetic effects of large current and charge densities in such a diode are not entirely understood. One effect is the screening process from the vacuum charges of the electric field produced by the diode.

B. DEFINITION OF SPACE CHARGE-LIMITED EMISSION

At high current densities, the electric field due to the electrons and/or plasma in the diode cannot be ignored.

Since the operation of these diodes is in the high current regime, this effect on the total electric field is important.

In a parallel-plate vacuum diode, the electric field produced by the diode is a function of gap distance and the potential difference applied. The vacuum electric field is constant and uniform. In this case, the current would be solely limited by the capacity of the cathode to emit charge. However, experiment shows that the current produced by a real diode is indeed finite, regardless of how much charge the cathode can emit. The reason for this is the decrease of the electric field due to the charge in the diode gap.

As electrons enter the diode, the charge creates a field opposite to the field applied by the diode. If emission from the cathode is allowed to continue, the electric field of the charges can eventually cancel the diode field. At this point, the charges emitted by the cathode are no longer accelerated away from the cathode and current is limited. This is space charge-limited emission. The current produced by the diode is not limited by the ability of the cathode to produce charge, but by the canceling effects the current has on the total electric field in the diode.

This limit has been explored since 1911. However, no closed analytic solution exists for finite diodes, which of course, all practical diodes are. Since ultimately, the power of an electromagnetic weapon is limited by the power produced by the electron beam, this limit is an important factor.

II. CHILD-LANGMUIR LAW FOR SPACE CHARGE-LIMITED EMISSION

A. GEOMETRY OF AN INFINITE VACUUM DIODE

The derivation of space charge-limited emission is a corner stone of plasma physics. Space charge-limited emission is defined as driving the electric field at a cathode emitter to zero, preventing any increase in the current density. The first form of the well-known Child-Langmuir Law was published in 1911. Beginning with Poisson's equation and the one-dimensional boundary conditions of a parallel infinite plate diode, the maximum current density can be found.

The geometry of the parallel infinite plate diode consists of an infinite grounded cathode and an anode held at some potential, V_o. The two plates of the diode are some distance, D, apart. Current is then allowed to flow across the vacuum gap. When the space charge-limited condition is met at the emission surface, the electric field produced by the current in the gap will balance the electric field produced by the potential difference of the two plates. This geometry is reflected in the following diagram.

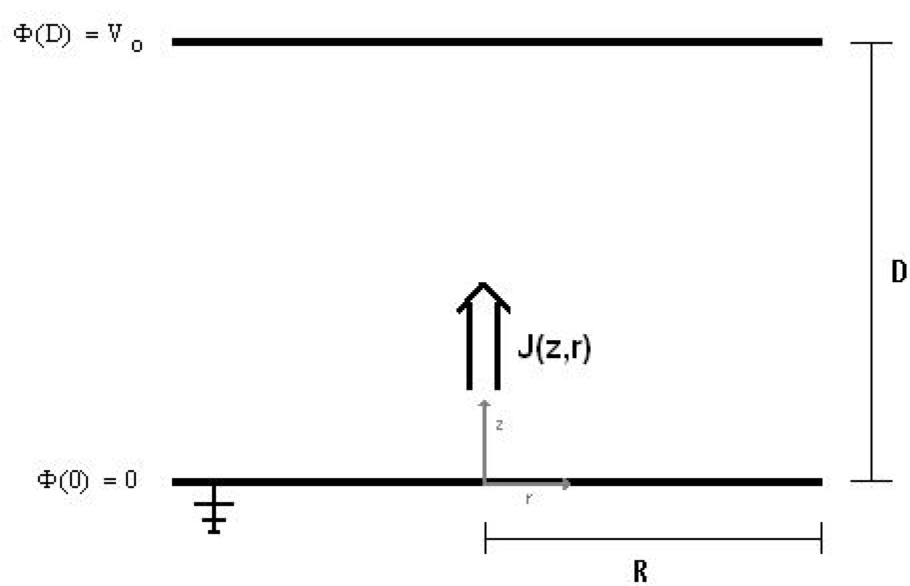

Figure 1. Diode Geometry

B. DERIVATION OF CHILD-LANGMUIR LAW

The unknown quantity is the maximum current density, J(z,r) that will be produced by such a diode. A reasonable starting point is the Poisson equation, relating the Laplacian of the potential, Φ, to the charge density, ρ.

$$\nabla^2\Phi(z,r) = -\frac{\rho(z,r)}{\varepsilon_o} \quad \textbf{(1a)}$$

For the one-dimensional case, the potential does not vary in the radial directional, so the Laplacian reduces to:

$$\frac{\partial^2\Phi(z)}{\partial z^2} = -\frac{\rho(z)}{\varepsilon_o} \quad \textbf{(1b)}$$

The current density is also related to the charge density and velocity of the charge by definition. Since no

charges are being created or destroyed in this steady-state problem, the relation is simply:

$$J(z) = \rho(z) * v(z) \qquad \textbf{(2)}$$

Then the charge density is expressed as:

$$\rho(z) = \frac{J(z)}{v(z)} \qquad \textbf{(3)}$$

The conservation of energy is used to express the velocity in terms of known parameters of the diode:

$$PE(0) + KE(0) = PE(z) + KE(z) \qquad \textbf{(4)}$$

Since the charge is assumed to be emitted with zero velocity, the kinetic energy at the cathode is zero. Since the cathode is grounded, the potential energy at the cathode is also zero, neglecting gravitation effects. Therefore, when expressions for the potential and kinetic energy of an electron are substituted:

$$0 = -e * \Phi(z) + \frac{1}{2} m_e * v(z)^2 \qquad \textbf{(5)}$$

Solving for velocity yields:

$$v(z) = \sqrt{\frac{2e}{m_e}} * \sqrt{\Phi(z)} \qquad \textbf{(6)}$$

Substitution of the expressions for charge density and velocity into the Poisson equation results in an expression relating potential and current density. The expression is a second-order non-linear differential equation.

$$\frac{\partial^2 \Phi(z)}{\partial z^2} = -\frac{J(z)}{\varepsilon_o} \sqrt{\frac{m_e}{2e}} * \frac{1}{\Phi(z)} \qquad \textbf{(7)}$$

The boundary conditions that lead to the space charge-limited solution are the potentials at the anode and the cathode, and the condition that the electric field at the cathode be zero.

$$\Phi(0) = 0 \qquad \textbf{(8a)}$$

$$\Phi(D) = V_o \tag{8b}$$

$$\frac{\partial \Phi(0)}{\partial z} = 0 \tag{8c}$$

The last boundary condition is imposed because we are solving for the space charge-limited emission case. Using the preceding boundary conditions, an outline of the solution to the differential equation follows. Multiplying both sides by the derivative of the potential and integrating:

$$\int \left(\frac{\partial \Phi}{\partial z} * \frac{\partial^2 \Phi}{\partial z^2} \right) = \frac{-J}{\varepsilon_o} \sqrt{\frac{m_e}{2e}} * \int \left(\frac{\partial \Phi}{\partial z} * \frac{1}{\sqrt{\Phi}} \right) \tag{9}$$

The integrals are of the form:

$$\int (u * du) = \int \left(u^{-\frac{1}{2}} * du \right) \tag{10}$$

Solving these integrals yields:

$$\frac{1}{2} * \left(\frac{\partial \Phi}{\partial z} \right)^2 = -\frac{J}{\varepsilon_o} \sqrt{\frac{m_e}{2e}} * 2 * \Phi^{\frac{1}{2}} + C \tag{11}$$

Here the constant of integration is zero from the boundary condition that the electric field is zero at the cathode surface. Taking the square root of both sides:

$$\frac{\partial \Phi}{\partial z} = -2 \sqrt{\frac{J}{\varepsilon_o}} \left(\frac{m_e}{2e} \right)^{1/4} * \Phi^{1/4} \tag{12}$$

Rearranging:

$$\Phi^{-1/4} \partial \Phi = 2 * \sqrt{\frac{J}{\varepsilon_o}} * \left(\frac{m_e}{2e} \right)^{1/4} * \partial z \tag{13}$$

Integrating this differential equation:

$$\frac{4}{3}\Phi^{3/4} = 2*\sqrt{\frac{J}{\varepsilon_o}}*\left(\frac{m_e}{2e}\right)^{1/4}*z+C \qquad \textbf{(14)}$$

Again, the boundary condition of the grounded cathode forces the constant to zero. By applying the anode boundary condition, one can solve for J, the Child-Langmuir Space Charge Limit for emission density:

$$J_{scl} = \frac{4}{9}\varepsilon_o\sqrt{\frac{2e}{m_e}}*\frac{V_o^{\ 3/2}}{D^2} \qquad \textbf{(15)}$$

This is the maximum current density that can be produce from a parallel infinite plate diode, given a potential difference, V_o, and spacing, D. Given this current density, substitution yields solutions to the following quantities:

Potential:

$$\Phi(z) = V_o*\left(\frac{z}{D}\right)^{4/3} \qquad \textbf{(16)}$$

Electric Field:

$$|E| = \frac{4}{3}*\frac{V_o}{D}*\left(\frac{z}{D}\right)^{1/3} \qquad \textbf{(17)}$$

Charge Density:

$$\rho(z) = \frac{4}{9}\varepsilon_o\frac{V_o}{D^2}*\left(\frac{z}{D}\right)^{-2/3} \qquad \textbf{(18)}$$

C. LIMITATIONS OF CHILD-LANGMUIR DERIVATION

Although the Child-Langmuir approach solves for the space charge limit, it is only valid for the one-dimensional case. It has been generalized to infinite cylindrical and spherical geometries, but it has not been solved for a finite emission area. Instead of approaching the problem from the Poisson equation, which generates a

non-linear differential equation, if the integral form of Coulomb’s law is used, the space charge limit can be developed by integration.

III. COULOMB'S LAW

A. COMPARING POISSON'S EQUATION WITH COULOMB'S LAW

In the diode configuration for the space charge limited formulation, it is assumed that the diode is in the steady state condition, and does not vary with time. With the addition of a background confining magnetic field, these conditions reduce Maxwell's Equations for electrodynamics down to Gauss's Law:

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_o} \qquad \textbf{(19)}$$

One result of Gauss's law is the Poisson equation, from which the Child-Langmuir solution for space charge-limited flow is derived. As a definition, the electric field is the negative of the gradient of the scalar potential:

$$\vec{E}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}) \qquad \textbf{(20)}$$

Using the definition of the Laplacian operator:

$$\vec{\nabla} \cdot (-\vec{\nabla}\Phi) = -\nabla^2\Phi \qquad \textbf{(21)}$$

Substitution into Gauss's Law gives the Poisson Equation:

$$\nabla^2\Phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_o} \qquad \textbf{(22)}$$

Gauss's Law also generates the integral form of Coulomb's law. Substituting the integral of a Dirac delta function into the expression for ρ:

$$\frac{\rho(\vec{r})}{\varepsilon_o} = \frac{1}{4\pi\varepsilon_o}\int 4\pi\delta^3(\vec{r} - \vec{r}\,')\rho(\vec{r}\,')d\tau' \qquad \textbf{(23)}$$

One form of the delta function is:

$$4\pi\delta^3(\vec{r}-\vec{r}') = \vec{\nabla}\cdot\left(\frac{(\vec{r}-\vec{r}')}{\left|(\vec{r}-\vec{r}')\right|^3}\right) \qquad \textbf{(24)}$$

Substituting for the Dirac delta function:

$$\frac{1}{4\pi\varepsilon_o}\int 4\pi\delta^3(\vec{r}-\vec{r}')\rho(\vec{r}')d\tau' = \frac{1}{4\pi\varepsilon_o}\int\vec{\nabla}\cdot\left(\frac{(\vec{r}-\vec{r}')}{\left|(\vec{r}-\vec{r}')\right|^3}\right)\rho(\vec{r}')d\tau' \qquad \textbf{(25)}$$

Since the integral in the above equation is with respect to $\vec{r}'$, and the divergence is with respect to $\vec{r}$, the divergence can be placed outside the integral, and the constant placed inside. From Gauss's law, this is equal to the divergence of the electric field:

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \vec{\nabla}\int\frac{1}{4\pi\varepsilon_o}\left(\frac{(\vec{r}-\vec{r}')}{\left|(\vec{r}-\vec{r}')\right|^3}\right)\rho(\vec{r}')d\tau' \qquad \textbf{(26)}$$

This must be hold for any $\vec{r}$, so the arguments must be equal. Thus, Coulomb's law is derived from Gauss's law:

$$\vec{E}(\vec{r}) = \int\frac{1}{4\pi\varepsilon_o}\left(\frac{(\vec{r}-\vec{r}')}{\left|(\vec{r}-\vec{r}')\right|^3}\right)\rho(\vec{r}')d\tau' \qquad \textbf{(27)}$$

Therefore, Coulomb's law and Poisson's equation are two different, but equally valid forms of Gauss's law. The results obtained by applying either equation to the conditions of the space charge-limited diode should be equivalent.

B. COULOMB'S LAW GEOMETRY

Coulomb's Law in integral form is an extension of Coulomb's force law for two point charges. Evaluating Coulomb's law results in an expression for the electric

field produced by a distribution of charges. The integral form is:

$$\vec{E}(\vec{r_f}) = \frac{1}{4\pi\varepsilon_o}\int \frac{1}{|\lambda|^2}\frac{\vec{\lambda}}{|\lambda|} dq(\vec{r_s}) \qquad \textbf{(28)}$$

In this case, the electric field can vary depending on the position where it is measured, the field point, $\vec{r_f}$. It is also dependent on the magnitude and position of the charge distribution. The vector $\vec{\lambda}$ is the displacement between the source point, $\vec{r_s}$, and the field point.

$$\vec{\lambda} = (\vec{r}_f - \vec{r_s}) \qquad \textbf{(29)}$$

The geometry is reflected in the following diagram.

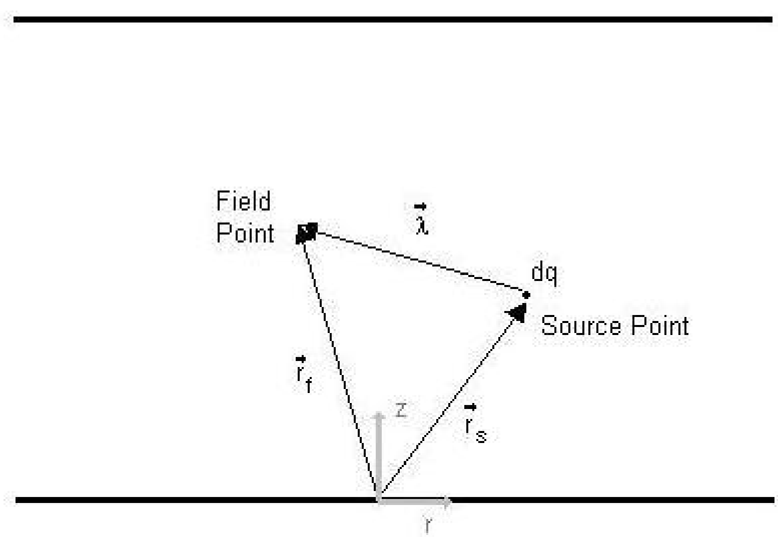

Figure 2. Coulomb Diode Geometry

The differential charge element, dq, can be expressed as the product of the charge density, ρ, and a differential volume element. In cylindrical coordinates, this becomes:

$$dq = \rho(\vec{r_s}) * r_s dr_s d\theta_s dz_s \qquad \textbf{(30)}$$

To simplify the analysis of Coulomb’s law, the beam can be assumed to be confined horizontally by an external

magnetic field, thus require only the analysis of the vertical component of the electric field. In this case:

$$\vec{\lambda}_z = (z_f - z_s)\vec{z} \quad \textbf{(31)}$$

Although the directionality of $\vec{\lambda}$ and $\vec{E}$ have radial and azimuthal dependence removed by the confinement assumption, it does not change the geometry of the problem. The magnitude of $\vec{\lambda}$ remains:

$$\left|\vec{\lambda}\right| = \sqrt{r_f^{\,2} + r_s^{\,2} - 2r_f r_s\left[\sin(\theta_f)\sin(\theta_s) + \cos(\theta_f)\cos(\theta_{s)}\right] + \left(z_f - z_s\right)^2} \quad \textbf{(32)}$$

Now that we have found expressions for all the components, the ability of software such as Mathcad to numerically solve integrals becomes attractive.

IV. MATHCAD WORKSHEET

A. CHILD-LANGMUIR

Mathcad is a commercially available numerical calculation program. Using Mathcad's symbolic interface, it is possible set up a worksheet that compares the electric field predicted by Child-Langmuir and from the integral form of Coulomb's law. If the two correspond, the flexibility of the worksheet will allow for the substitution for various charge distribution geometries.

The first inputs into the worksheet are the global variables that control the initial geometry of the problem. For the initial purpose of comparing Child-Langmuir and Coulomb's Law, the gap distance, D, and potential difference, V_o, are constant factors, and can be set to unity to simplify the analysis. The same is true of the permittivity of free space, ε_o.

The geometry of the Child-Langmuir diode is two infinite plates. The corresponding parameter is an infinite radius, R.

For comparison the results of the Child-Langmuir solution are defined. The functional form of the potential, $\Phi(z)$, is given. Then the Child-Langmuir electric field, Ecl(z), is the one-dimensional divergence, or the derivative with respect to z. The vector component of the electric field in the z-direction is denoted by a subscript z. $Ecl_z(z)$ is plotted for comparison with the electric field calculated by Coulomb's law.

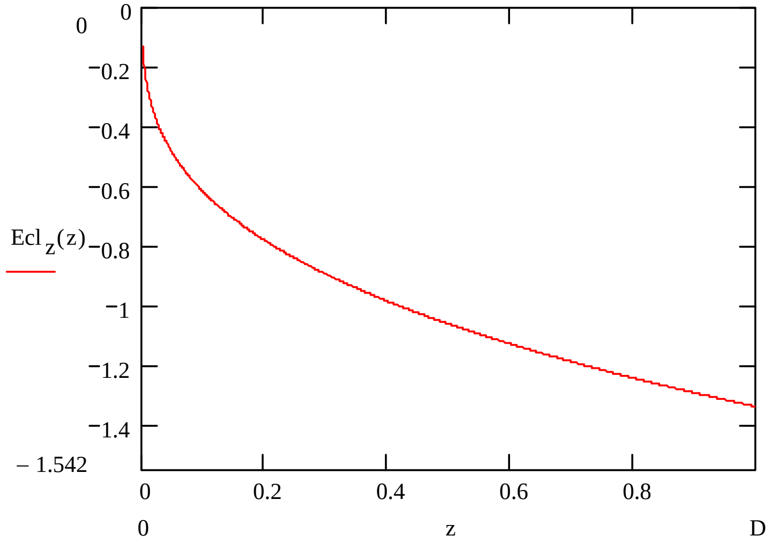

Figure 3. Child-Langmuir Electric Field

B. COULOMB'S LAW

The charge density follows as the divergence of Ecl(z). This charge density will be used in the Coulomb's law integral. However, a distinction must be made be in this transition between the source point, z_s, and the field point, z_f. In Child-Langmuir, the electric field is evaluated at the same z as the charge density. For Coulomb's law, the charge density is integrated over all the sources to give the field at a distinct point.

The Coulomb integral is integrated over all θ_s, therefore the choice of θ_f is arbitrary. Choosing θ_f to be zero simplifies the expression for the displacement vector. The magnitude of the displacement vector becomes:

$$\left|\vec{\lambda}\right| = \sqrt{r_f{}^2 + r_s{}^2 - 2r_f r_s\left[\cos(\theta_s)\right] + \left(z_f - z_s\right)^2} \qquad \textbf{(33)}$$

As a first check to the model, the conditions of the Child-Langmuir diode are applied. The symmetry of the infinite diode allows r_f to go to zero. This further simplifies the displacement vector, removing azimuthal dependence.

$$\left|\vec{\lambda}\right| = \sqrt{r_s{}^2 + \left(z_f - z_s\right)^2} \qquad \textbf{(34)}$$

The evaluation of the Coulomb integral can be plotted as a function of z_s.

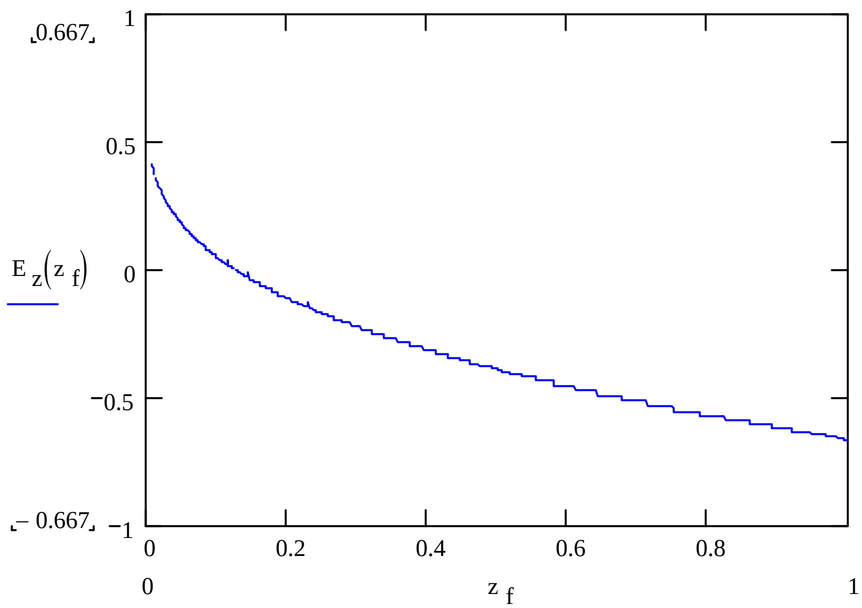

Figure 4. Coulomb Electric Field

C. COMPARING COULOMB'S LAW AND CHILD-LANGMUIR

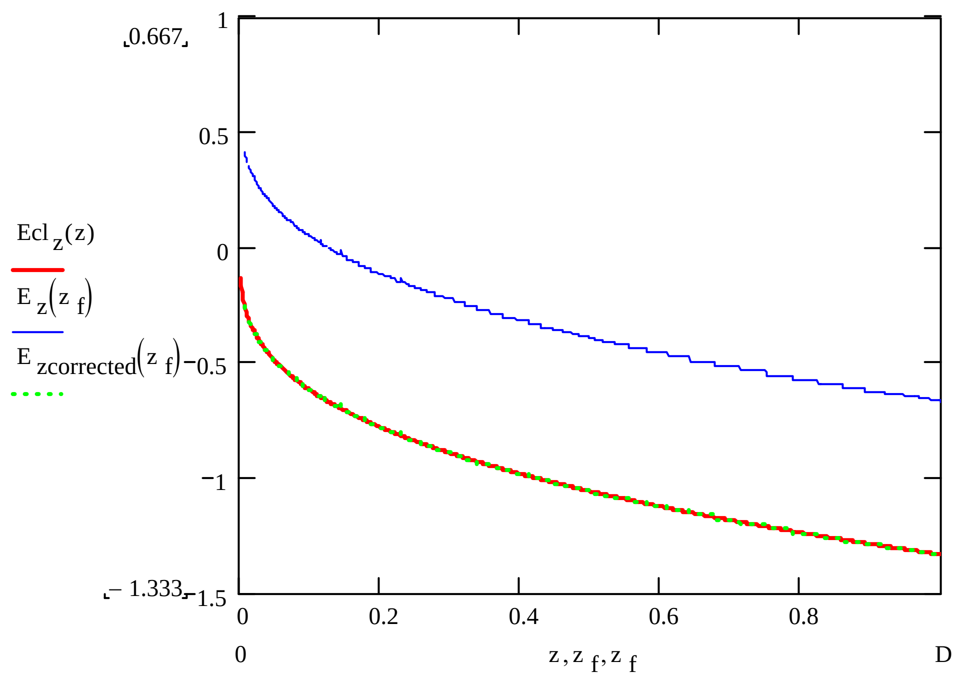

Figure 5. Comparison of Child-Langmuir and Coulomb's Law

It is evident from the plot that the limits of Coulomb field differ from the limits of the Child-Langmuir field. Both the upper and lower limit are too high by $\frac{2}{3}\frac{V}{D}$. Subtracting $\frac{2}{3}\frac{V}{D}$ from the entire Coulomb field, it is seen that the Coulomb field differs from the Child-Langmuir field by this constant.

The above plot shows that the Coulomb's law integral and the Child-Langmuir solution differ only by a constant.

D. ADJUSTING THE COULOMB FIELD

That the Coulomb integral differs should not be a surprising result. The integral only accounts for the effect of the charge in the diode gap. However, the total electric field, as predicted by Child-Langmuir, is also influenced by the potential difference of the conducting plates.

One approach to finding the effect of the conducting plates is the method of images. However, the solving for the images of a charge distribution on two conductors that are held at different potentials proved beyond the scope of this thesis.

Another approach is to apply the boundary condition of the potential difference across the gap.

$$V_o = -\int_0^D E_{zcorrected}(z_f)dz_f \qquad \textbf{(35)}$$

Here, $E_{zcorrected}(z_f)$ is a sum of the electric field from the space charge, and a correction factor, the electric field produced by the conducting cathode and anode.

$$E_{zcorrected}(z_f) = E_z(z_f) + E_{zplates}(z_f) \qquad \textbf{(36)}$$

Integrating the space charge electric field:

$$\int_0^D E_z(z_f)dz_f = -\frac{1}{3}V_o \quad \textbf{(37)}$$

Subtracting from the total potential difference:

$$\int_0^D E_{zplates}(z_f)dz_f = -\frac{2}{3}V_o \quad \textbf{(38)}$$

From this, the correction to electric field from the plates is:

$$E_{zplates}(z_f) = -\frac{2}{3}\frac{V_o}{D} \quad \textbf{(39)}$$

This is exactly the magnitude of the difference of the Coulomb field from Child-Langmuir. This formulation shows a correspondence between the predictions of Child-Langmuir's solving of a differential equation and integration using Coulomb's Law.

E. GENERALIZING THE COULOMB INTEGRAL

This integral formulation has the same limitations as the differential equation solution, in that it only holds for an infinite area diode. To be effective for solving for a finite diode, the formulation must account for off-axis values of r_f, introducing an azimuthal dependence.

The generalized Coulomb integral does not have simple symmetries to allow for contraction of a dimension and is:

$$E_z(z_f) = \int_0^D \int_0^R \int_0^{2\pi} \frac{\rho(z_s) * \dfrac{(z_f - z_s)}{\sqrt{r_f^{\,2} + r_s^{\,2} - 2r_f r_s[\cos(\theta_s)] + (z_f - z_s)^2}}}{\left(\sqrt{r_f^{\,2} + r_s^{\,2} - 2r_f r_s[\cos(\theta_s)] + (z_f - z_s)^2}\right)^2} * r_s d\theta_s dr_s dz_s \quad \textbf{(40)}$$

In this form, Mathcad is unable to reliably solve for $E_z(z_f)$ over the range $0 < z_f < D$.

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V. CONCLUSIONS AND FUTURE WORK

The goal of this thesis was to show a correspondence between the space charge limited conditions predicted from the Child-Langmuir law, and the conditions predicted from Coulomb’s law. In this respect, the Mathcad worksheet was successful. However, it should be noted, that assumptions imposed on the Coulomb integral limit the worksheet to the one-dimensional, infinite diode case.

The assumptions that limit the Mathcad worksheet are assuming azimuthal symmetry of the charge distribution about the field point. For a finite distribution, this constrains the field point to be on the centerline of the distribution. This limitation does not allow for the exploration of the edge effects of a finite beam, where the maximum current density exceeds the Child-Langmuir limit. These “wings” could vastly affect the performance of practical vacuum diode, as much more power would be produce than that predicted by the simple infinite approximation.

Opportunity for future work exists in extending the worksheet to allow Mathcad to solve for arbitrary geometries. Including the azimuthal dependence would generalize the Coulomb integral, making the worksheet a powerful tool for exploring the electric field of a vacuum diode.

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APPENDIX

The following pages are a rich text copy of the Mathcad worksheet used.

Global

$R \equiv \infty$

Subscripts on λ and E represent vector

$D \equiv 1$

$V_o \equiv 1$

$\varepsilon_o \equiv 1$

$\delta \equiv .001$

Child-Langmuir:

Potential:

$$\Phi(z) := V_o \cdot \left(\frac{z}{D}\right)^{\frac{4}{3}}$$

Electric Field:

$$Ecl_z(z) := \frac{d}{dz} - \Phi(z)$$

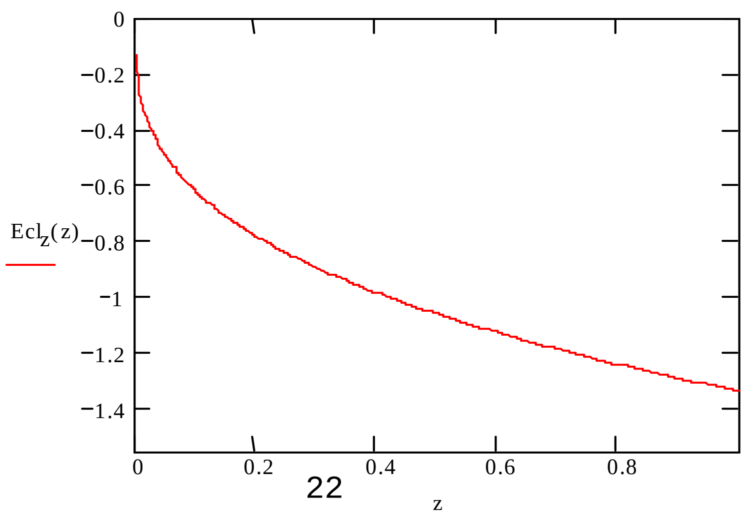
$$Ecl_z(z) \rightarrow \frac{-4}{3} \cdot z^{\frac{1}{3}}$$

Charge Density:

$$\rho(z_s) := \varepsilon_o \cdot \frac{d}{dz_s} Ecl_z(z_s)$$

$$\rho(z_s) \rightarrow \frac{-4}{9 \cdot z_s^{\frac{2}{3}}}$$

Graph of the Electric Field:



Coulomb's Law:

Infinite Diode Case:

Displacement Vector:

$$\lambda_z(z_f, z_s, r_s) := \frac{(z_f - z_s)}{\sqrt{(r_s)^2 + (z_f - z_s)^2}}$$

$$\lambda(z_f, z_s, r_s) := \sqrt{(r_s)^2 + (z_f - z_s)^2}$$

Charge Density:

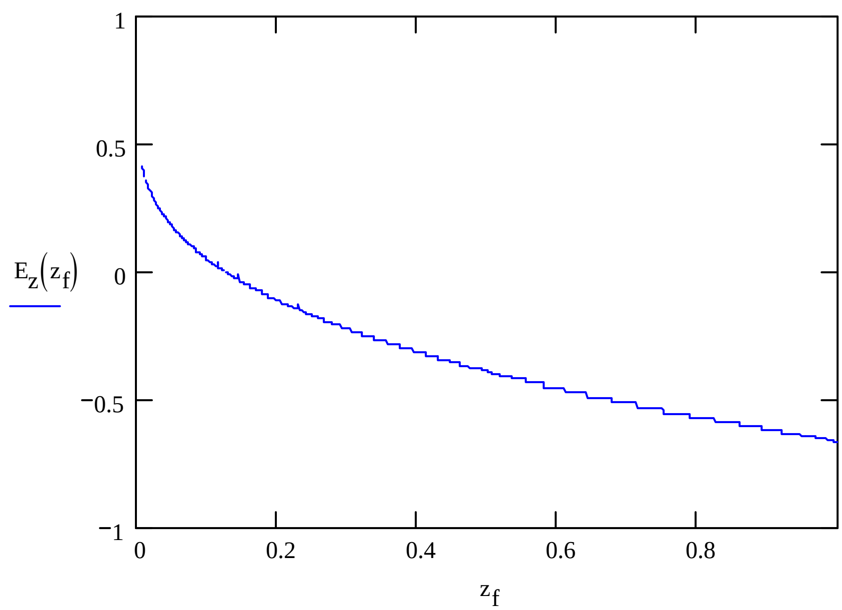
$$\rho(z_s) := \frac{-4}{9 \cdot z_s^{\frac{2}{3}}}$$

Electric Field:

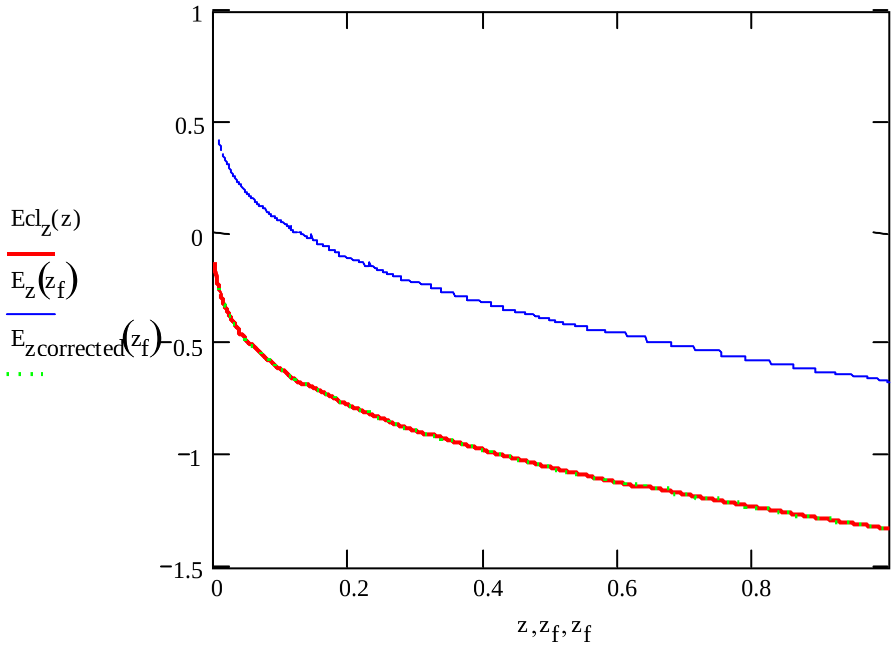
θ dependence removed due to azimuthal symmetry.

$$E_z(z_f) := \left(\frac{1}{2 \cdot \varepsilon_o}\right) \cdot \int_0^D \int_0^R \left(\frac{\lambda_z(z_f, z_s, r_s) \cdot \rho(z_s) \cdot r_s}{\lambda(z_f, z_s, r_s)^2}\right) dr_s \, dz_s$$

$E_z(.0003) = 0.581$



$$E_{zcorrected}(z_f) := E_z(z_f) - \frac{2 \cdot V}{3 \cdot D}$$



Space Charge Effect on Diode:

$$V := \int_{.0101}^{D} E_z(z_f)\, d\, z_f$$

$V = -0.338$

Allow for a Finite Diode:

Displacement Vector:

$$\lambda(r_f, z_f, z_s, r_s, \theta_s) := \sqrt{\left[r_f^{\,2} + r_s^{\,2} - 2r_f \cdot r_s \cdot (\cos(\theta_s))\right] + (z_f - z_s)^2}$$

$$\lambda_z(r_f, z_f, r_s, z_s, \theta_s) := \frac{(z_f - z_s)}{\sqrt{\left[r_f^{\,2} + r_s^{\,2} - 2r_f \cdot r_s \cdot (\cos(\theta_s))\right] + (z_f - z_s)^2}}$$

z-component of the Electric Field:

$$E_z(r_f, z_f) := \left(\frac{1}{4 \cdot 3.14 \cdot \varepsilon_o}\right) \cdot \int_0^D \left[\left[\int_0^R \left(\int_0^{2\cdot\pi} \frac{\rho(z_s) \cdot \lambda_z(r_f, z_f, r_s, z_s, \theta_s) \cdot r_s}{\lambda(r_f, z_f, z_s, r_s, \theta_s)^2}\, d\theta_s\right) dr_s\right]\right] dz_s$$

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